Definitions of Order

Definition of "*Big-O*"

For a given complexity function $f(n)$, $O(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constant c and some nonnegative integer N such that for all $n \geq N$,

$$g(n) \leq c \times f(n).$$

Definition of Ω

For a given complexity function $f(n)$, $\Omega(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constant c and some nonnegative integer N such that for all $n \geq N$,

$$g(n) \geq c \times f(n).$$

Definition of Θ

For a given complexity function $f(n)$,

$$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n)).$$

This means that $\Theta(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constants c and d and some nonnegative integer N such that for all $n \geq N$,

$$c \times f(n) \leq g(n) \leq d \times f(n).$$

Definition of "*Little-o*"

For a given complexity function $f(n)$, $o(f(n))$ is the set of complexity functions $g(n)$ satisfying the following: For every positive real constant c there exists some nonnegative integer N such that for all $n \geq N$,

$$g(n) \leq c \times f(n).$$

Properties of Order

1. $g(n) \in O(f(n))$ if and only if $f(n) \in \Omega(g(n))$.
2. $g(n) \in \Theta(f(n))$ if and only if $f(n) \in \Theta(g(n))$.
3. If $b > 1$ and $a > 1$, then $\log_a n \in \Theta(\log_b n)$. In other words all logarithmic complexity functions are in the same complexity category, referred to simply as $\Theta(\lg n)$.
4. If $0 < a < b$, then $a^n \in o(b^n)$.
5. For all $a > 0$, $a^n \in o(n!)$.
6. Consider the following ordering of complexity categories:

   $$\Theta(\lg_n) \quad \Theta(n) \quad \Theta(n \lg n) \quad \Theta(n^2) \quad \Theta(n^j) \quad \Theta(n^k) \quad \Theta(a^n) \quad \Theta(b^n) \quad \Theta(n!)$$

   where $k > j > 2$ and $a > b > 1$. If a complexity function $g(n)$ is in a category that is to the left of the category containing $f(n)$, then

   $$g(n) \in o(f(n)).$$
7. If $f_1(n) \in O(g(n))$ and $f_2(n) \in O(h(n))$, then $f_1(n) + f_2(n) \in O(g(n) + h(n))$. Less formally, this means $f_1(n) + f_2(n) \in \max[O(g(n)), O(h(n))]$.
8. If $f_1(n) \in O(g(n))$ and $f_2(n) \in O(h(n))$, then $f_1(n) * f_2(n) \in O(g(n) * h(n))$.
9. If $f(n)$ is a polynomial of degree k, then $f(n) \in \Theta(n^k)$.
10. For any constant k, $\lg^k(n) \in O(n)$.